Randomized Algorithms
CSCI 432

Matrix Multiplication Verification

5	8	1
3	6	7
4	1	5

A

X

2	4	7
5	1	2
3	6	5

B

=

53	34	56
57	60	68
28	47	55

C

Problem: Given three $n \times n$ matrices A, B, and C, verify that $AB = C$.

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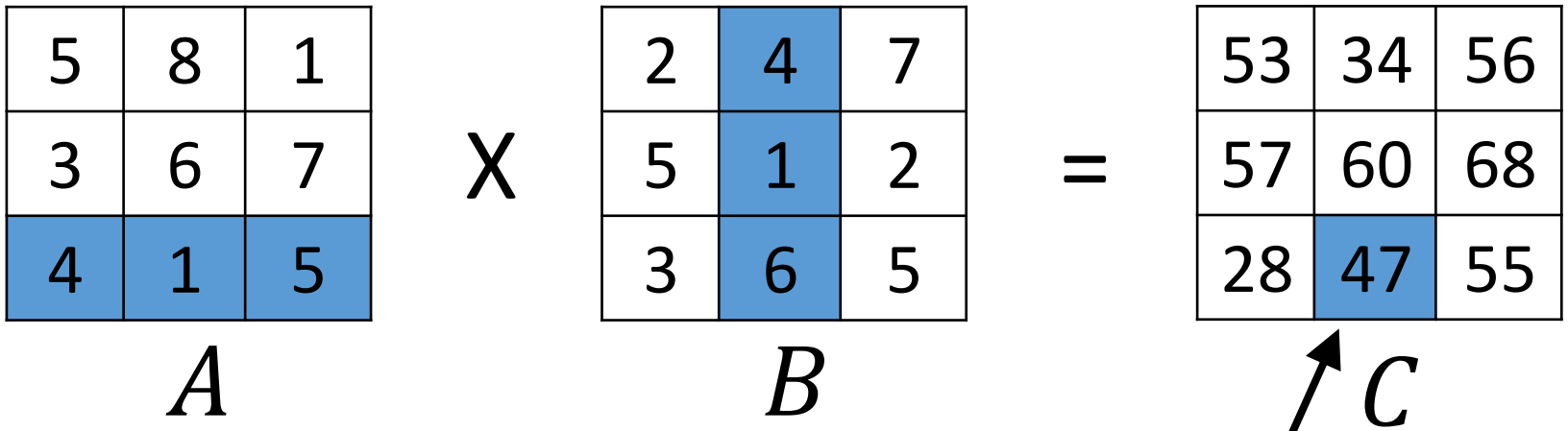
Running Time?

Matrix Multiplication



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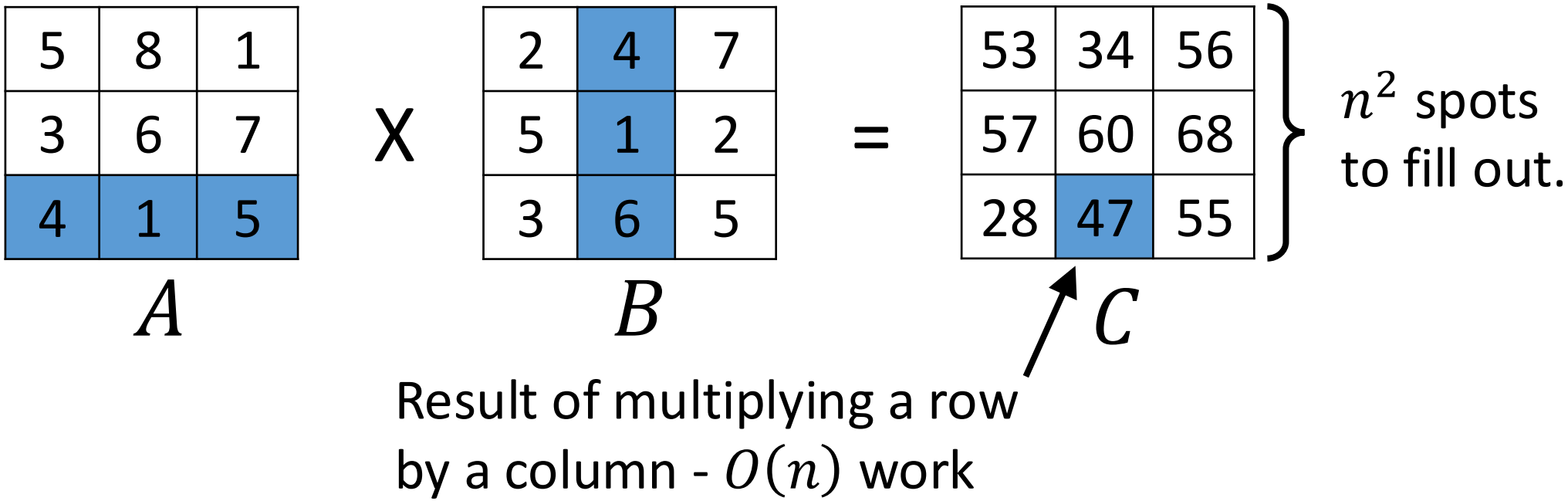

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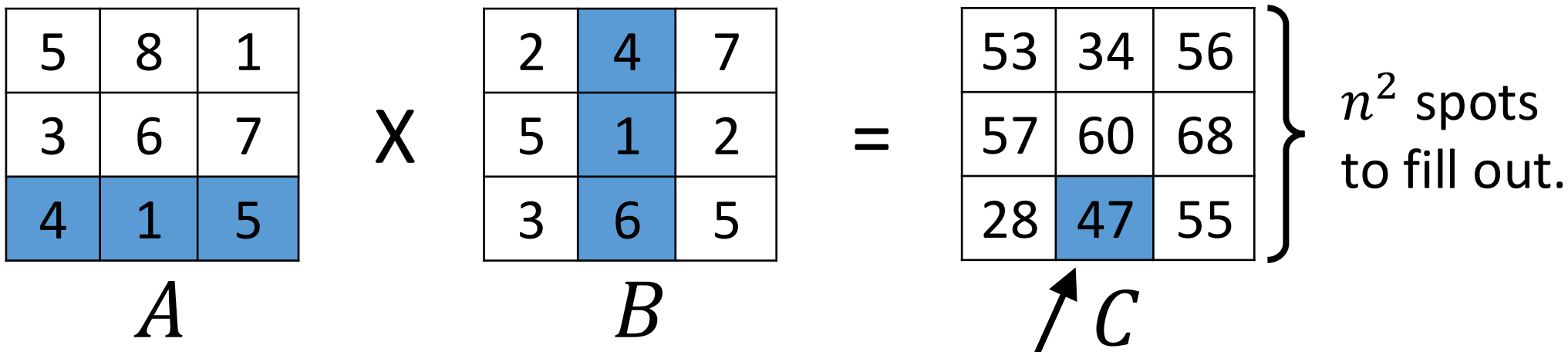

Thus, $O(n^3)$ total running time.

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Freivalds' Algorithm (on input A, B, C):

1.

2.

3.

Freivalds' Algorithm

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C

$r =$

1
0
1

Freivalds' Algorithm (on input A, B, C):

1. Generate $n \times 1$ vector r containing random 0 or 1 values.
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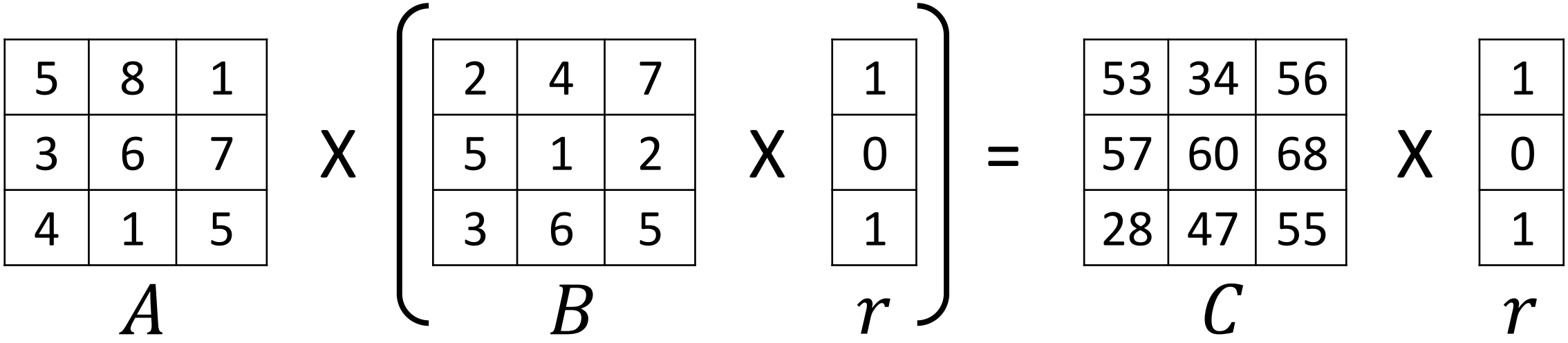
Freivalds' Algorithm



$$r = \begin{array}{|c|}\hline 1 \\ \hline 0 \\ \hline 1 \\ \hline \end{array}$$

Freivalds' Algorithm (on input A, B, C):

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2. Check if $A(Br) = Cr$.
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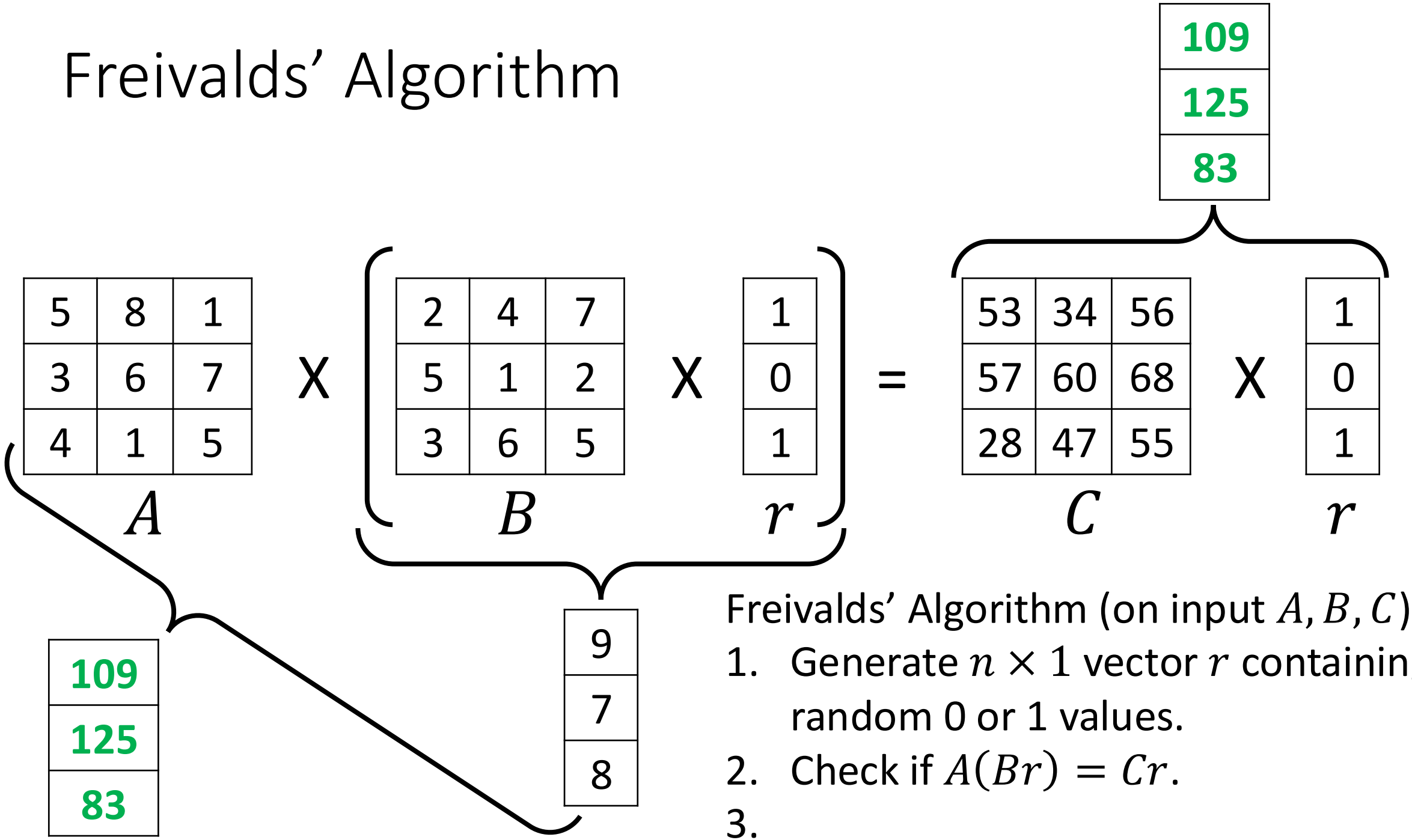
Freivalds' Algorithm



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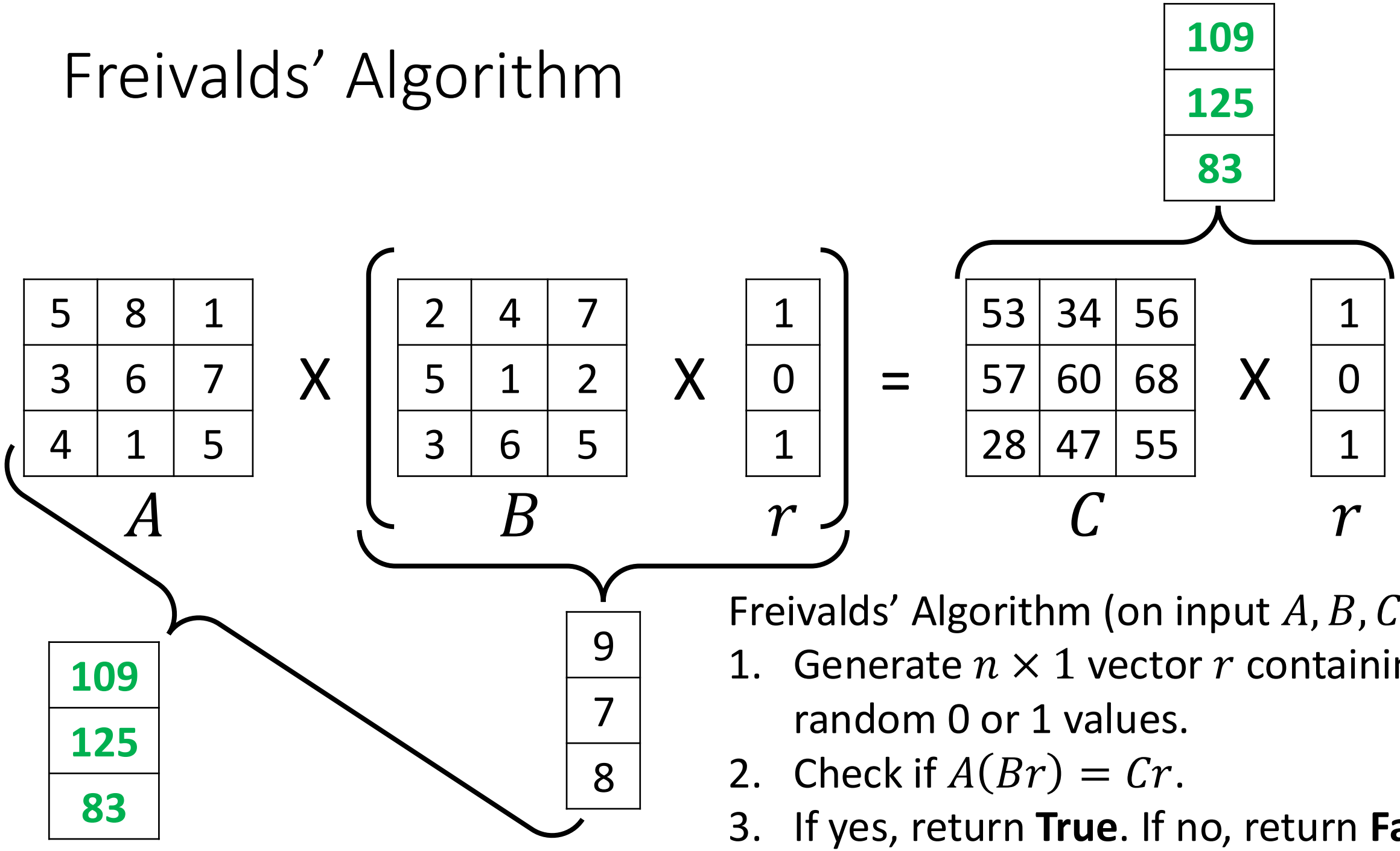

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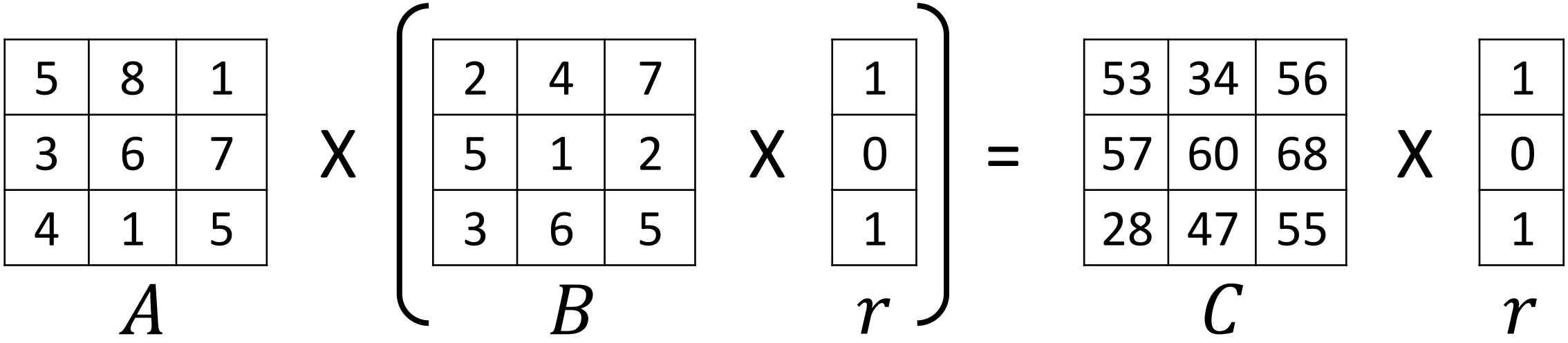

$A(Br) = (AB)r$, so if $AB = C$, then $A(Br) = (AB)r = Cr$

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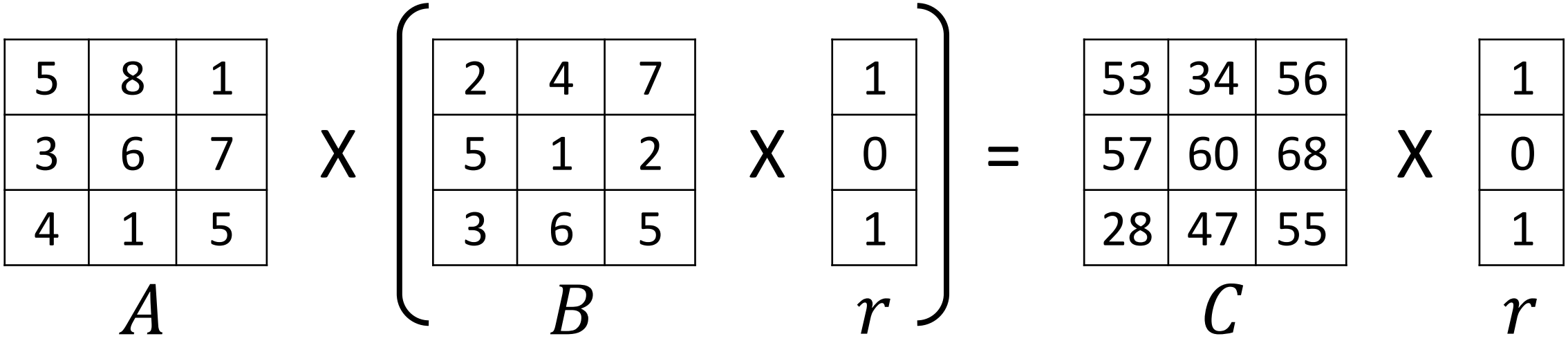

Randomness (generally) impacts an algorithm's correctness or running time.

	Correctness	Running Time
Monte Carlo	Random	Bounded
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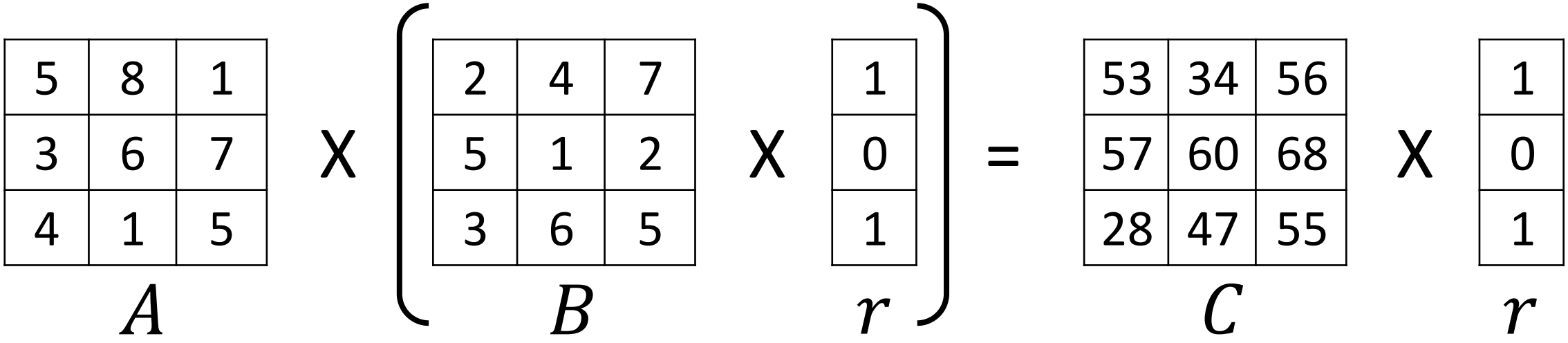

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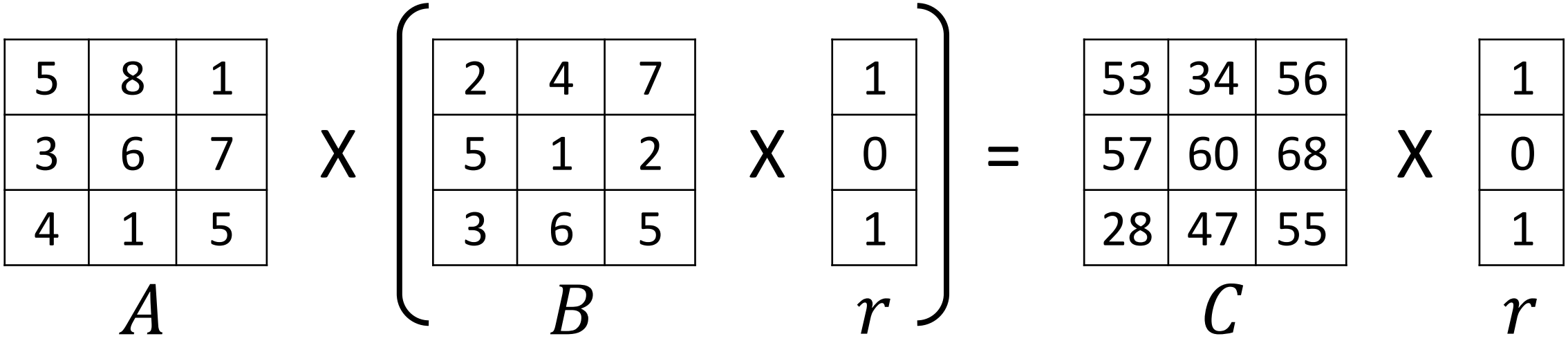

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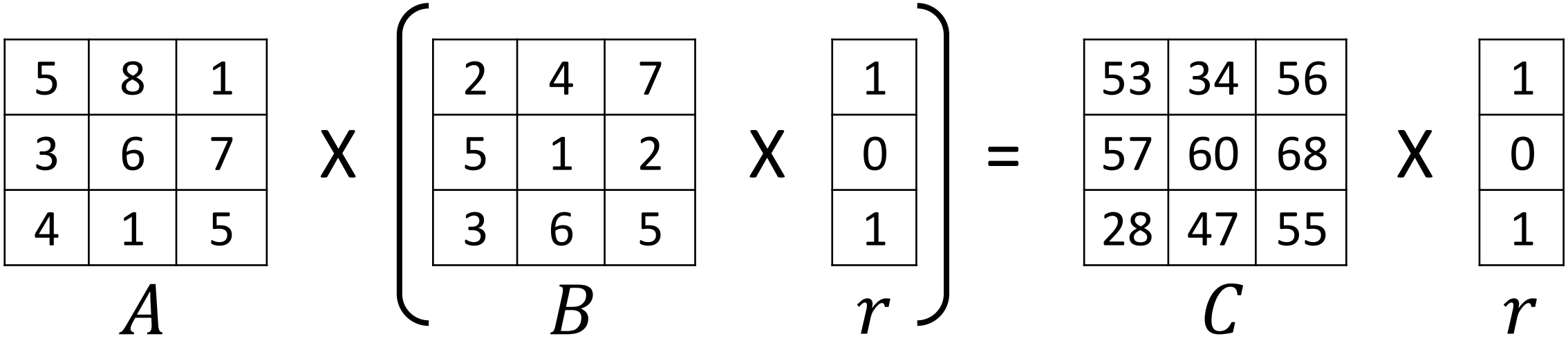

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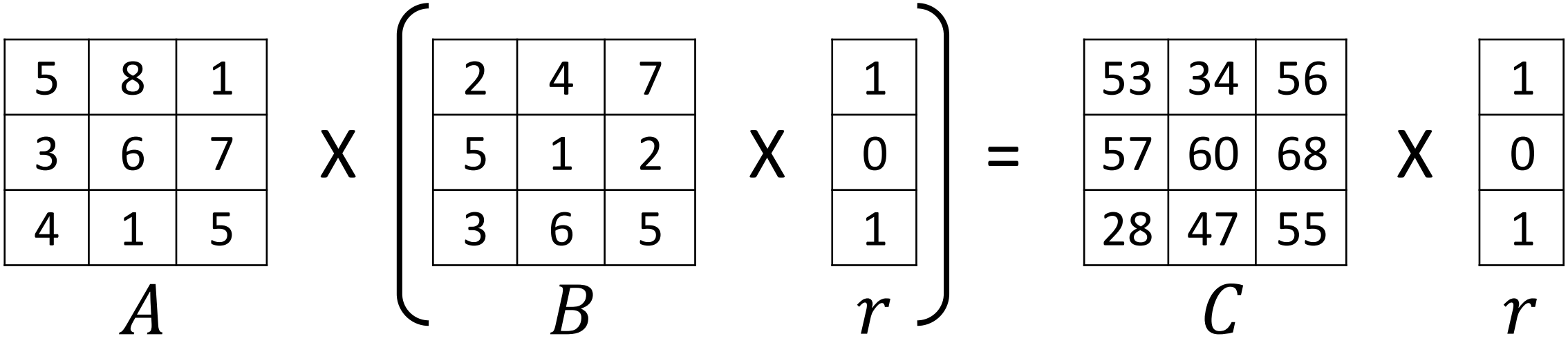

Running Time?

$$A(Br) = Cr$$

$O(n^2)$ $O(n^2)$ $O(n^2)$

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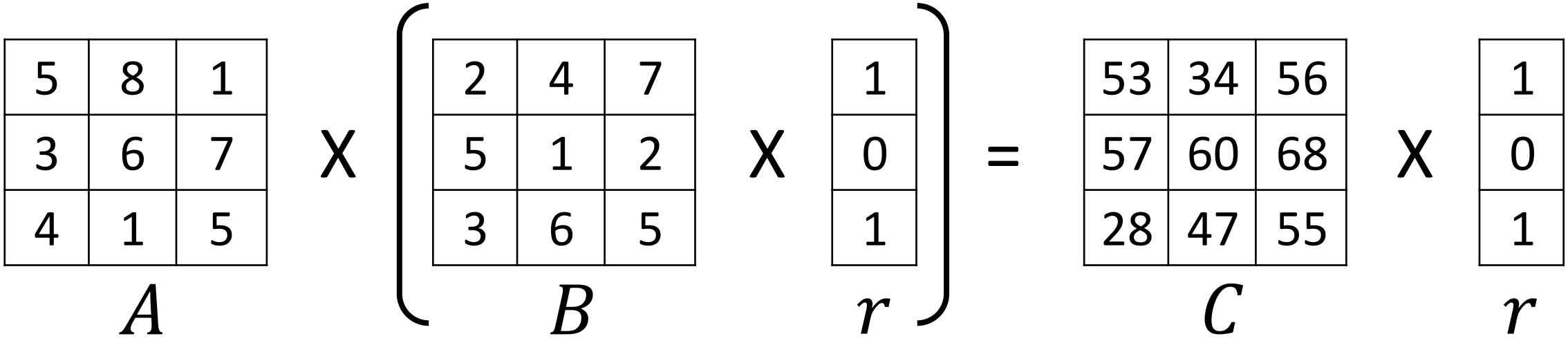

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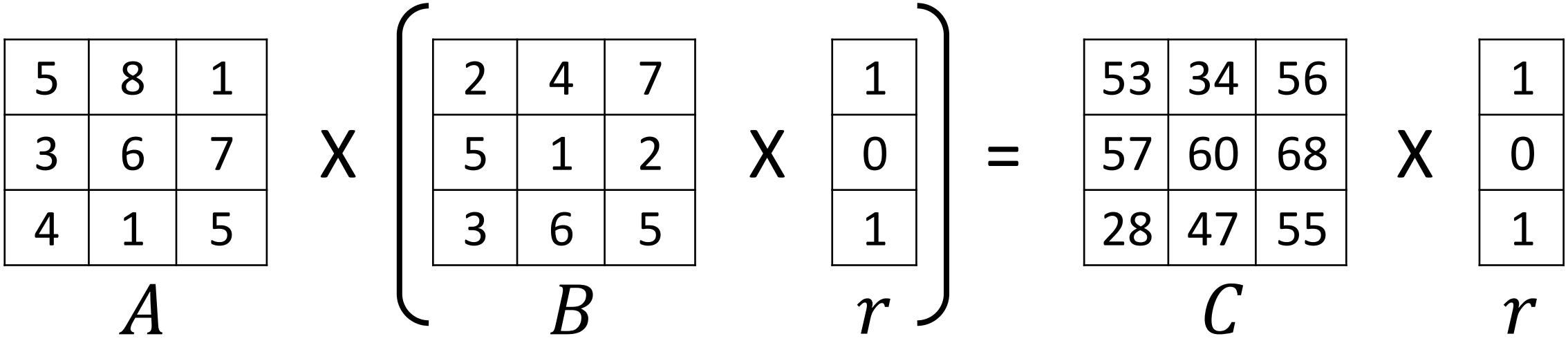

Running Time?

$$(AB)r = Cr$$

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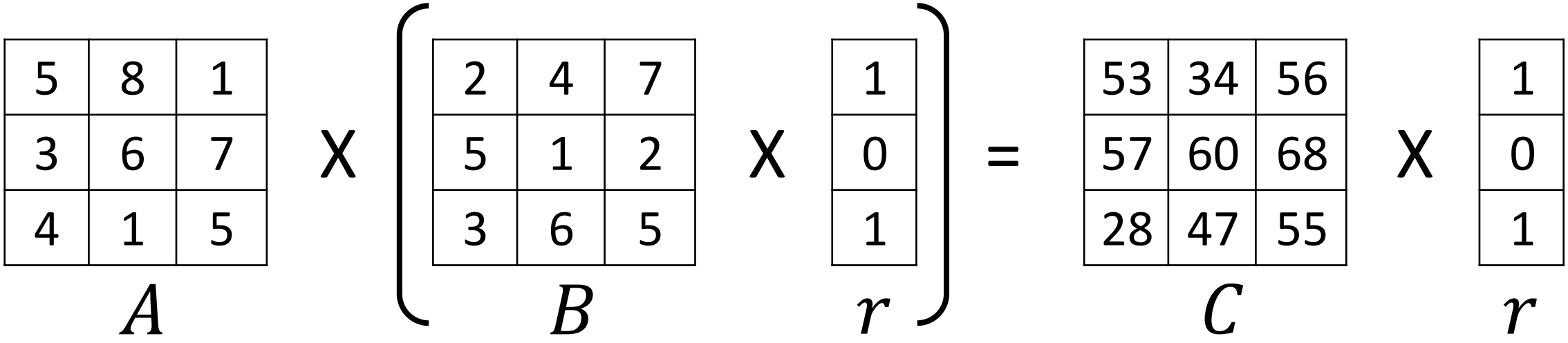

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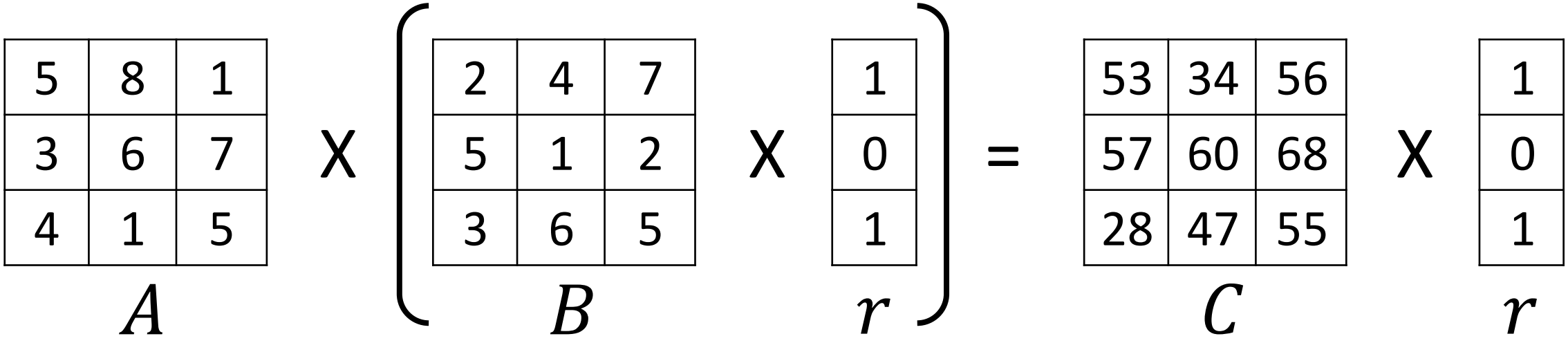

Randomized Algorithm Error Analysis:

- Where is the error?
- What conditions lead to error?
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- How do we mitigate the error?

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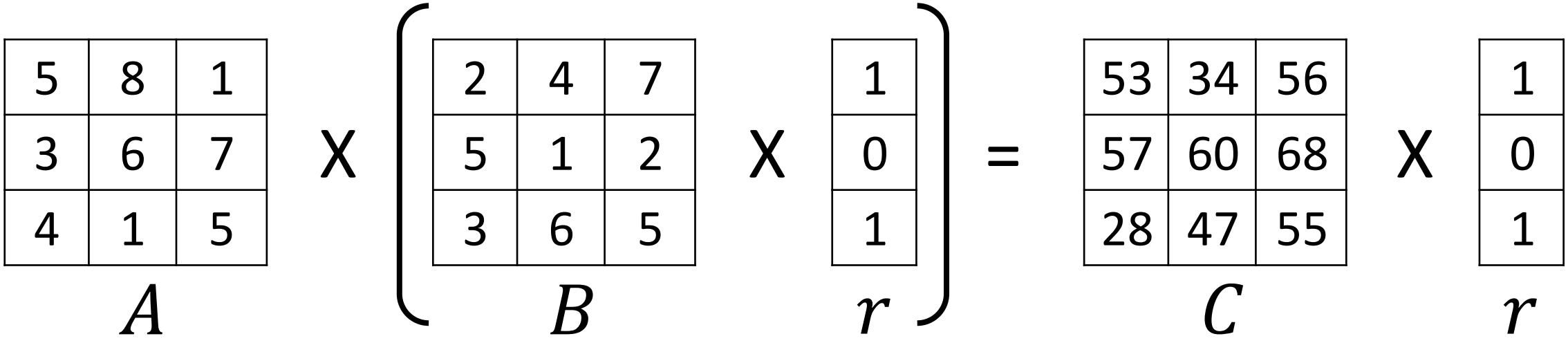

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1. False Negative.

2. False Positive.

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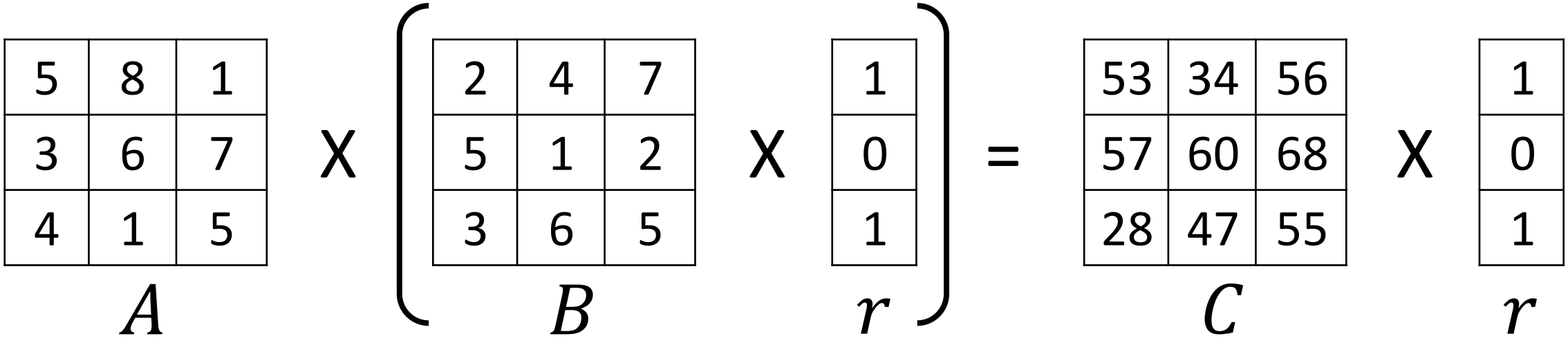

Error Types:

1. False Negative. $AB = C$, but Freivalds' says **false**.
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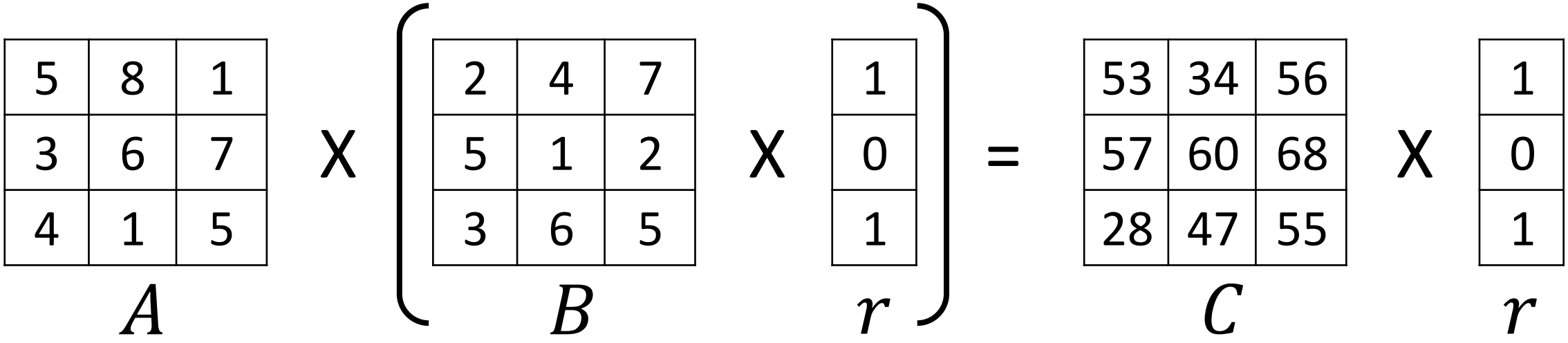

Error Types:

1. False Negative. $AB = C$, but Freivalds' says **false**.
2. False Positive. $AB \neq C$, but Freivalds' says **true**.

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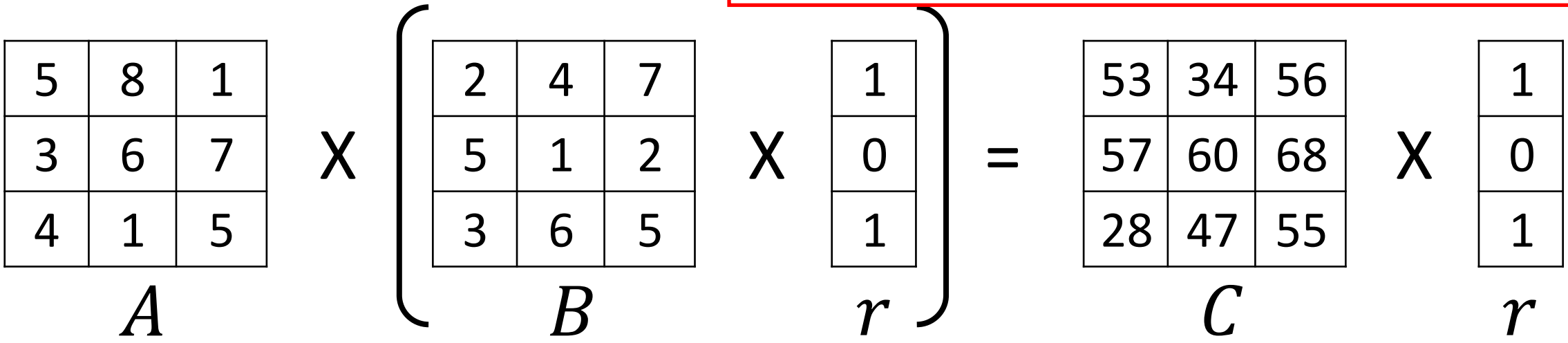
Freivalds' Algorithm

What is the probability of a false negative?



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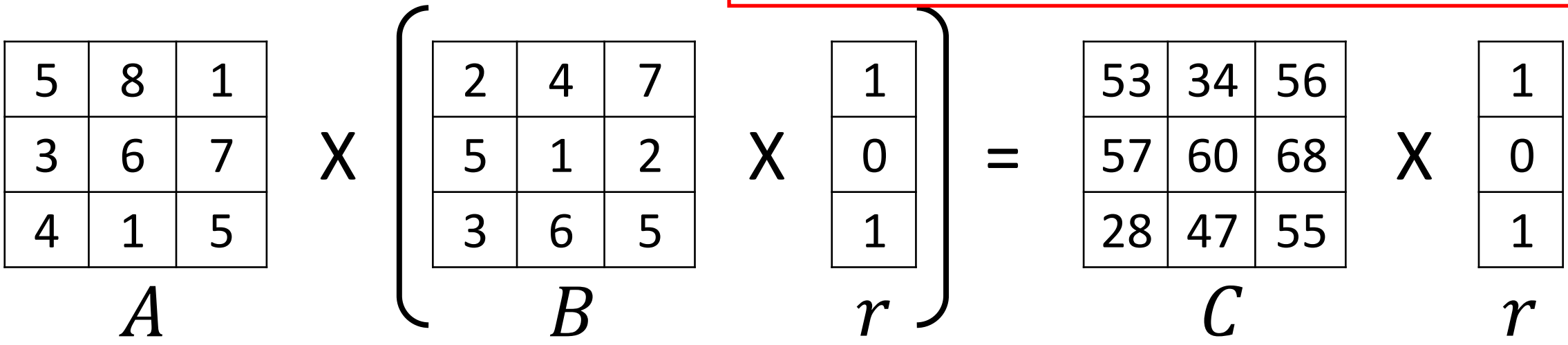
What is the probability of a false negative?

Zero. If $AB = C$, then $(AB)r = Cr$ and $A(B$ Cr.



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Freivalds' Algorithm

What is the probability of a false negative?

Zero. If $AB = C$, then $(AB)r = Cr$ and $A(B$ Cr.

If $AB = C$, Freivalds' never returns false.

$$\begin{bmatrix} 5 & 8 & 1 \\ 3 & 6 & 7 \\ 4 & 1 & 5 \end{bmatrix} \quad A$$

$$X \left(\begin{bmatrix} 2 & 4 & 7 \\ 5 & 1 & 2 \\ 3 & 6 & 5 \end{bmatrix} \quad B \quad X \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad r \right) =$$

$$\begin{bmatrix} 53 & 34 & 56 \\ 57 & 60 & 68 \\ 28 & 47 & 55 \end{bmatrix} \quad C \quad X \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \quad r$$

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1. ~~False Negative. $AB = C$, but Freivalds' says **false**~~
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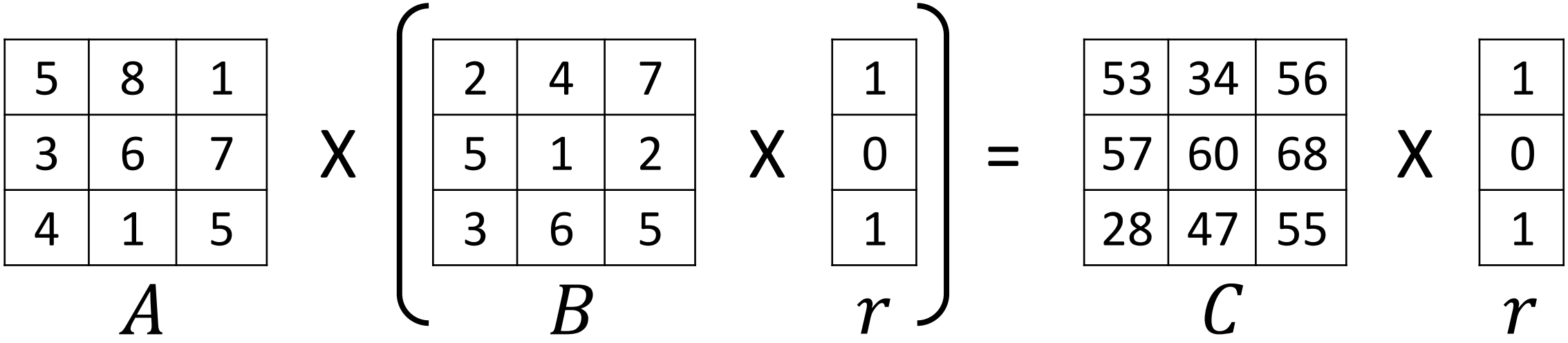
Freivalds' Algorithm

Is the probability of a false positive also 0?



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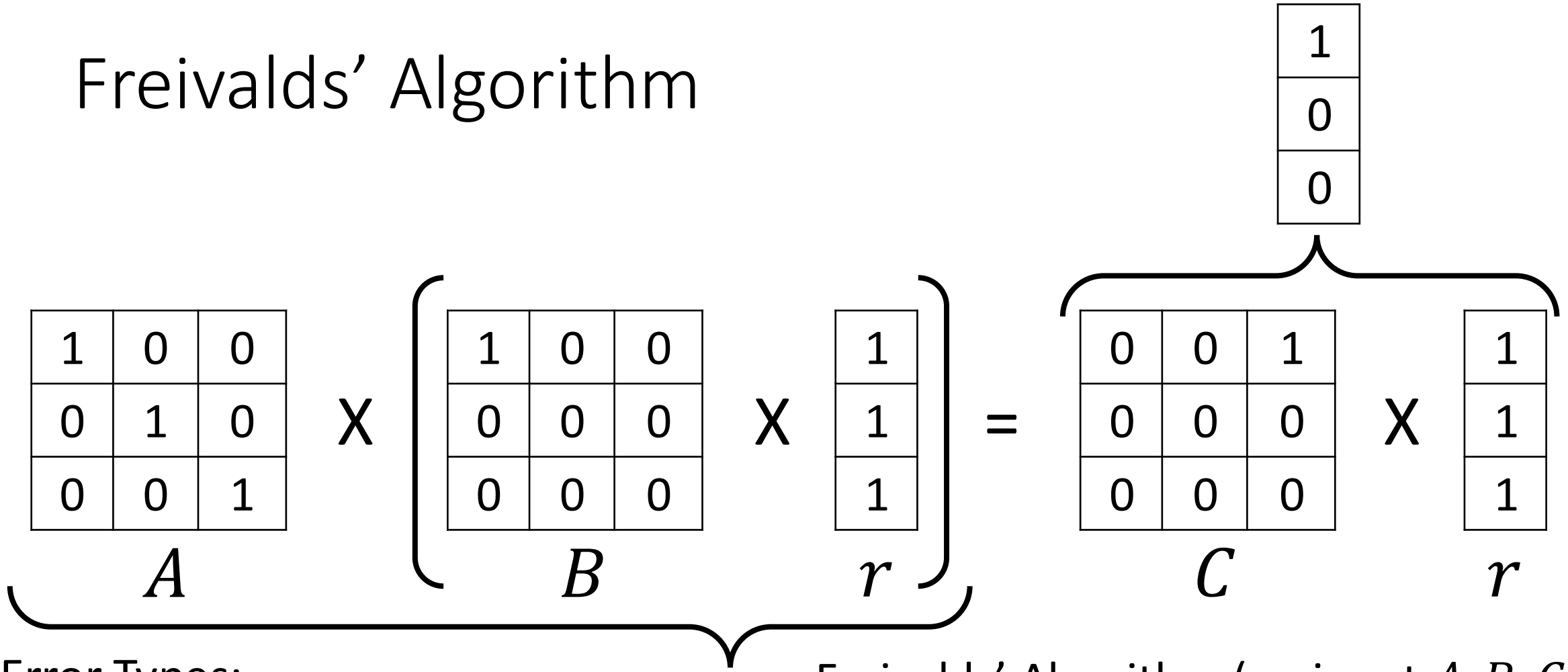

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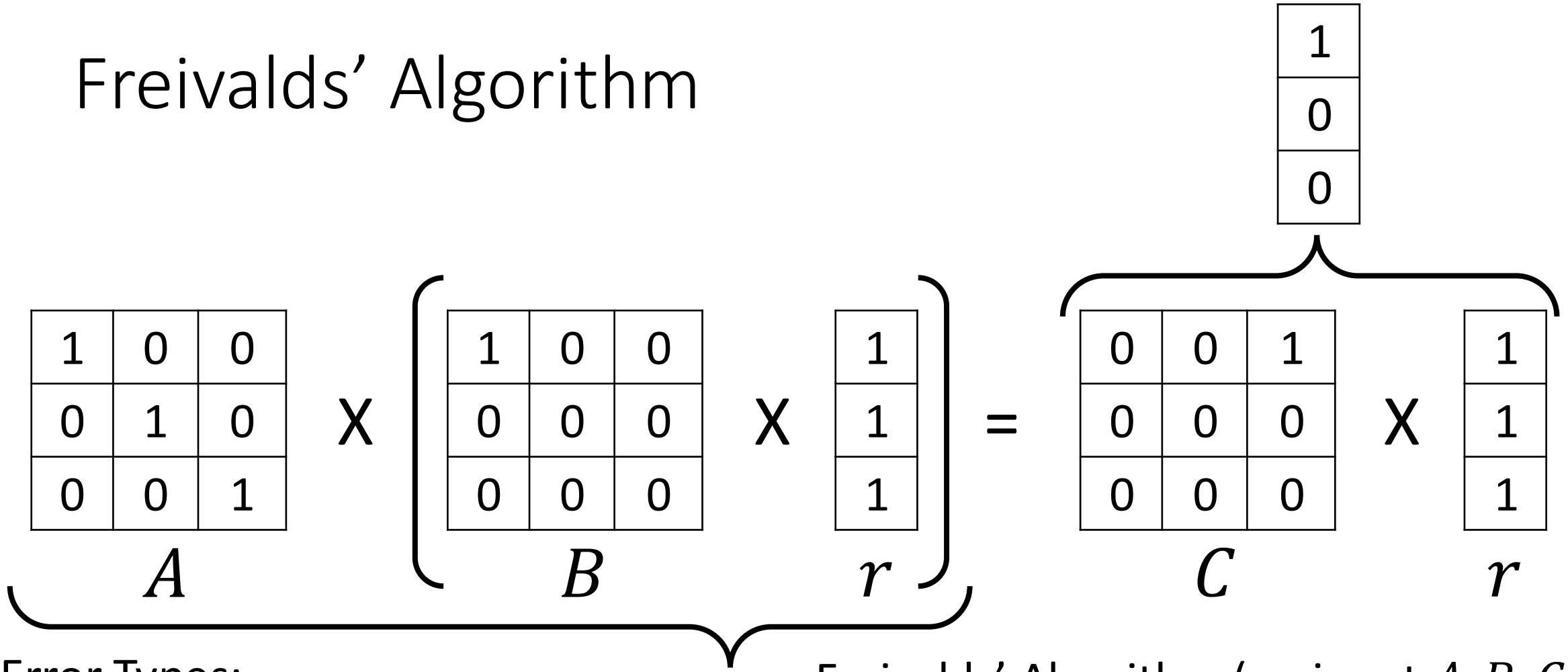

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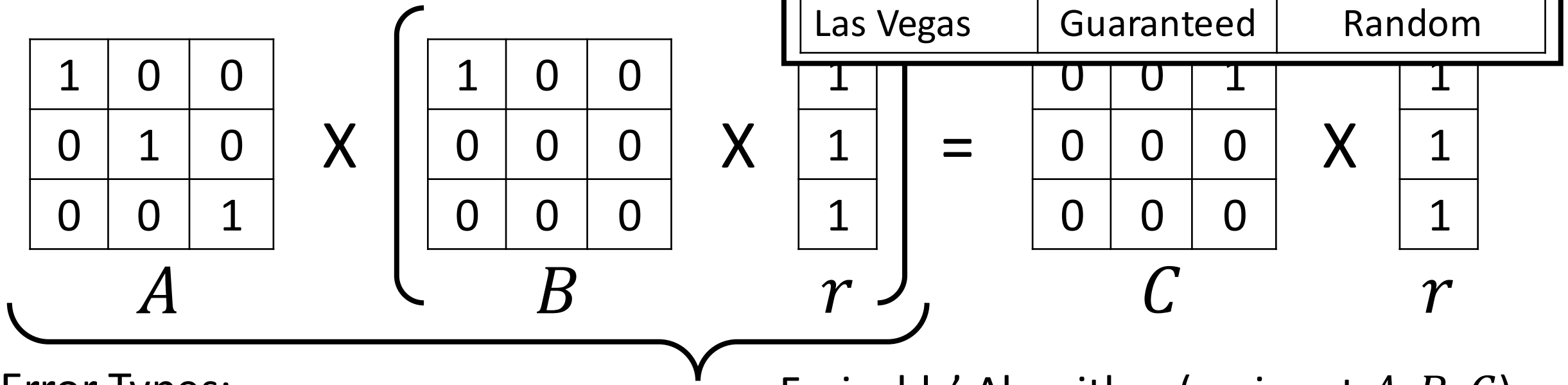

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1
0
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- Where is the error?
- What conditions lead to the error?
- How likely is the error?
- How do we mitigate the error?

Freivalds' Error Analysis

- Where is the error?

  $AB \neq C$, but Freivalds' returns **true**.

- What conditions lead to the error?

- How likely is the error?

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Freivalds' Error Analysis

- Where is the error?
  $AB \neq C$, but Freivalds' returns **true**.
- What conditions lead to the error?
  $AB \neq C$, but $ABr = Cr$.
- How likely is the error?

- How do we mitigate the error?

Freivalds' Error Analysis

- Where is the error?
  $AB \neq C$, but Freivalds' returns **true**.
- What conditions lead to the error?
  $AB - C \neq 0$, but $ABr = Cr$.
- How likely is the error?

- How do we mitigate the error?

Freivalds' Error Analysis

- Where is the error?
  $AB \neq C$, but Freivalds' returns **true**.
- What conditions lead to the error?
  $AB - C \neq 0$, but $(AB - C)r = 0$.
- How likely is the error?

- How do we mitigate the error?

Freivalds' Error Analysis

Freivalds' fails when:

$AB - C \neq 0$ and $(AB - C)r = 0$

Freivalds' Error Analysis

Freivalds' fails when:

$$AB - C \neq 0 \text{ and } (AB - C)r = 0$$

i.e., if $D = AB - C$, when $D \neq 0$ and $Dr = 0$

Freivalds' Error Analysis

What is the probability that Freivalds' fails?

Freivalds' fails when:

$$AB - C \neq 0 \text{ and } (AB - C)r = 0$$

i.e., if $D = AB - C$, when $D \neq 0$ and $Dr = 0$

Conditional Probability:

$\Pr[E|F]$ - Probability of event E occurring given that event F has occurred.

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What is the probability that Freivalds' fails?

$\Pr[\text{Freivalds' returns } \mathbf{true} | AB \neq C]$ ← Given that the instance is wrong, what is the chance that Freivalds' says it is correct?

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i.e., if $D = AB - C$, when $D \neq 0$ and $Dr = 0$

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$\Pr[\text{Freivalds' returns } \mathbf{true} | AB \neq C]$ ⟵ Given that the instance is wrong, what is the chance that Freivalds' says it is correct?

$\Pr[AB \neq C | \text{Freivalds' returns } \mathbf{true}]$ ⟵ Given that Frieval's says the instance is correct, what is the chance it is actually wrong?

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$$\Pr[\text{Freivalds' returns } \textbf{true} | AB \neq C] = \Pr[Dr = 0 | D \neq 0] = ?$$

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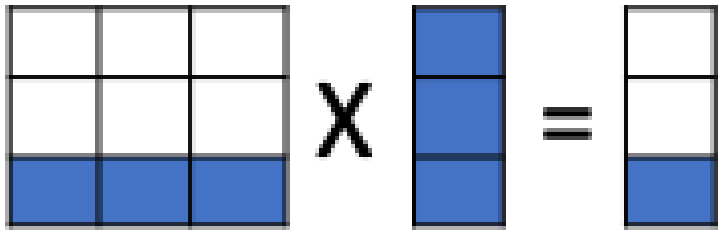

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Freivalds' Algorithm (on input A, B, C):

1. Generate $n \times 1$ vector r containing random 0 or 1 values.
2. Check if $A(Br) = Cr$.
3. If yes, return **True**. If no, return **False**.

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